- The Fourier transform X of the signal x is referred to as the frequency spectrum of x.
- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the magnitude spectrum of x.
- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the phase spectrum of x.
- Since the Fourier transform is a function of a real variable, a signal can potentially have information at any real frequency.
- Earlier, we saw that for periodic signals, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is *usually represented using two plots*, one showing the magnitude spectrum and one showing the phase spectrum.

- Recall that, for a *real* signal x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is *conjugate symmetric*), which is equivalent to

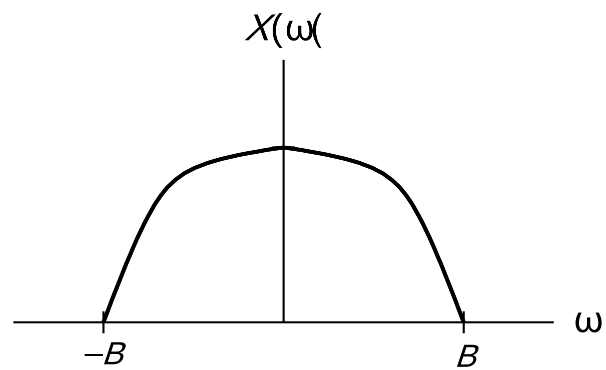
$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a *real* signal is always *even*.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a *real* signal is always *odd*.
- Due to the symmetry in the frequency spectra of real signals, we typically *ignore negative frequencies* when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

- A signal x with Fourier transform X is said to be bandlimited if, for some nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- In the context of real signals, we usually refer to B as the bandwidth of the signal x.
- The (real) signal with the Fourier transform X shown below has bandwidth B.



- One can show that a signal *cannot be both time limited and bandlimited*. (This follows from the time/frequency scaling property of the Fourier transform(.

Section 5.5

Fourier Transform and LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega.($$

- The function H is called the frequency response of the system.
- A LTI system is *completely characterized* by its frequency response H. The
- above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the magnitude response of the system.
- The quantity $\arg H(\omega)$ is called the phase response of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

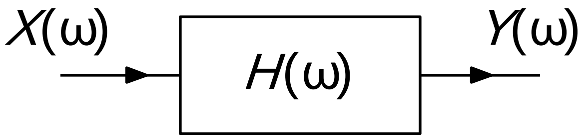
- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is *real*, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is *even* and the phase response $\arg H(\omega)$ is *odd*).

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.



- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

Frequency Response and Differential Equation

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t) \quad \text{where} \quad M \leq N.$$

- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a *rational function*.

Section 5.6

Application: Circuit Analysis

- A resistor is a circuit element that opposes the flow of electric current. A
- resistor with resistance R is governed by the relationship

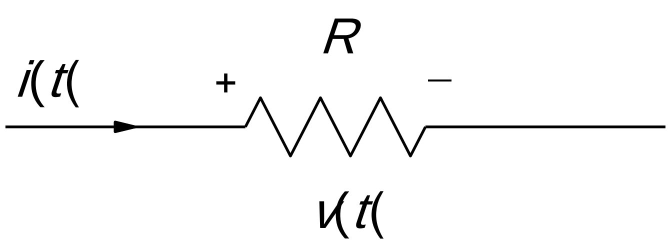
$$v(t) = Ri(t) \quad \text{or equivalently,} \quad i(t) = \frac{1}{R}v(t) \quad ,$$

where v and i respectively denote the voltage across and current through the resistor as a function of time.

- In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \quad \text{or equivalently,} \quad I(\omega) = \frac{1}{R}V(\omega) \quad ,$$

where V and I denote the Fourier transforms of v and i, respectively. In
- circuit diagrams, a resistor is denoted by the symbol shown below.



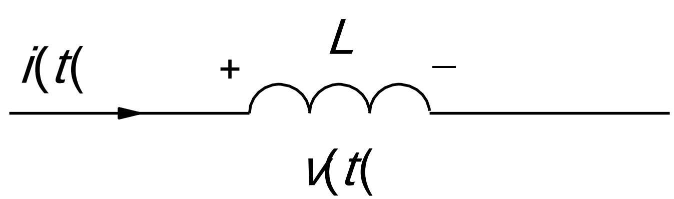
- An inductor is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor with inductance L is governed by the relationship

$$v(t) = L\frac{d}{dt}i(t) \quad \text{or equivalently,} \quad i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau ,$$

where v and i respectively denote the voltage across and current through the inductor as a function of time.
- In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega L I(\omega) \quad \text{or equivalently,} \quad I(\omega) = \frac{1}{j\omega L}V(\omega)$$

where V and I denote the Fourier transforms of v and i, respectively. In
- circuit diagrams, an inductor is denoted by the symbol shown below.



- A capacitor is a circuit element that stores electric charge.
- A capacitor with capacitance C is governed by the relationship

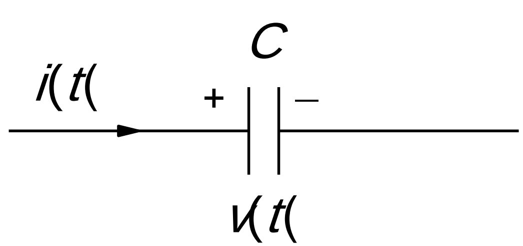
$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \quad \text{or equivalently,} \quad i(t) = C\frac{d}{dt}v(t),$$

where v and i respectively denote the voltage across and current through the capacitor as a function of time.

- In the frequency domain, the above relationship becomes

$$V(\omega) = \left(\frac{1}{j\omega C}\right) I(\omega) \quad \left(\text{or equivalently, } I(\omega) = j\omega C V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively. In
- circuit diagrams, a capacitor is denoted by the symbol shown below.



- The Fourier transform is a very useful tool for circuit analysis.
- The utility of the Fourier transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.

Section 5.7

Application: Filtering